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Fixed Point Results for Non-Commuting Mappings in Metric Space via W-Distance

Abstract

In this paper, some unique common fixed point results are provedfor non-commuting JSR and JSR^*mappings in the complete metric space viaw-distance. In support of the results some examples are also given.

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Archana Agrawal
Research Scholar,
Deptt. of Mathematics and Computer Science,
U.T.D, R. D. University,
Jabalpur (M. P.), India

Manoj Kumar Shukla
Professor,
Deptt. of Mathematics,
Institutefor Excellence in Higher Education,
Bhopal, (M.P.)

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Introduction

The well-known Banach contraction principle, which declares thaton a complete metric space foreach single-valued contraction self-mapping, always there exists a unique fixed point. This basic principle has been exploited and generalized by many researchers using different contraction conditions, applying different mappings in different spaces.

Review of Literature

Nadler [8] has used the concept of Hausdorff metric and obtained a multi-valued version of the Banach contraction principle .Among others Husain and Latif [2], Feng and Liu [1] have generalized Nadler's fixed point result without using the Hausdorff metric.On the other hand, Kannan [4] has proved an interesting fixed point result for single-valued maps in the setting of metric spaces which is not an extension of the Banach contraction principle .While Latif and Beg [6] have obtained a multivalued version of Kannan's fixed point result. In 1996 the team of Kada, Suzuki and Takahashi [3] came with a new and more generalized concept of w-distance hence, many earlier results are improved. Simultaneously Suzuki and Takahashi [12] worked on weakly contractive maps for single and multi-valued functions and produced some important generalizations of Banach contraction principle and based Nadler's results. Parallel to this work there co-researchers Suzuki [13] improve the Kannan's fixed point results by using w-distance. After that a bulk of investigations have been observed [5] [10][13] and [15].

Applying the concept of w-distence Abdul Latif et al. [7]proved some fixed point and common fixed point results for multi-valued maps with the setting of metric spaces, by which they generalized and improve many results including the results of Latif and Beg [6], Suzuki [13], Kannan [4].

Till then no work is reported in this field.

Aim of the Study

Our aimis to consider non-commuting JSR and JSR^*mappings with w-distance in complete metric space and proved unique common fixed point results.We have furnish some examples in support of our main results.

2. Preliminaries

A bulk of literature exist with commuting and non-commuting mappings.We are defining non-commuting pair of maps as JSR and JSR^* maps which is more improved than the known mappings.

On a metric space the concept of w-distance was introduced by Kada et al.[3] in the following manner:

Let $p: X \times X \to [0, \infty)$ be a function over a metric space (X, d), then$p$ is called w- distance if

1. $\forall\ x, y, z \in \mathrm{X},\ p(x,z) \le p(x,y) + p(y,z)$

2. $\forall\ x \in X$ and $y_n \to y$ in X, $p(x,y) \leq \liminf p(x,y_n)$, that is p is lower semi continuous with respect to the second variable y
3. for any given $\varepsilon > 0$, there must be a $\delta > 0$ such that $p(x,z) \leq \delta$ and $p(z,y) \leq \delta \Longrightarrow p(x,y) \leq \varepsilon$

Clearly, everymetric is a *w*-distance but not conversely.

Definition 2.1

A pair (S,T) of self-mappings S and T ona metric space$(X,d)$is said to beweakly commuting if and only if

$d(STx\ ,TSx) \leq d(Sx,Tx)$for each x in X.

Definition 2.2

Let S and Tbe theself-mappings ona metric space (X,d) with a w-distance p,then$S$ and$T$ are said to be p-weakly commuting if and only if

$max\,[p(STx,TSx),p(TSx\ ,STx)\,] \leq p(Sx\ ,Tx)$ for each x in X.

Definition 2.3

Let S and Tbe theself-mappings ona metric space (X,d) .Then S and T are said to be weakly compatible if and only if each sequence $\{x_n\}$ such that for some$t$ in X.

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \mathrm{t} \implies \lim_{n\to\infty} d(STx_n\ \ TSx_n) = 0$$

Definition 2.4

Let S and T be the self-mappings ona metric space $(X,d)$with w-distance p,then S and T are said to be (p) compatible if every sequence $\{x_n\}$ such that for some t in X

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ as } n \to \infty$$

$$\Longrightarrow max\,[p(STx_n,TSx_n),p(TSx_n\ \ ,STx_n)] \geq 0, \text{as } n \to \infty$$

Definition 2.5

The pair(S,T) of two self-mappings S and Ton a metric space(X,d) is said to be S-JSR mappings if and only if each sequence $\{\mathrm{x_n}\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \mathrm{t} \text{ for some } t \text{ in } X$$

$$\Longrightarrow \alpha d(STx_n\ \ ,Tx_n\ ) \leq\ \alpha d(SSx_n,Sx_n)$$

where$\alpha = \limsup$ or $\liminf$.

Definition 2.6

The pair(S,T) of two self-mappings S and T on a metric space (X,d) is said to be S-$JSR_{(p)}$ mappings if and only if each sequence $\{\mathrm{x_n}\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \text{ in } X$$

$$\Longrightarrow \max\{\alpha p(STx_n,Tx_n), \alpha p\ (Tx_n,STx_n)\}$$
$$\leq \max\{\alpha p(Sx_n,Sx_n), \alpha p(Sx_n\ ,SSx_n)\}$$
$$\text{where } \alpha = \lim\sup \text{ or } \lim\inf$$

Definition 2.7

The pair(S,T) of two self-mappings S and T on a metric space (X,d) is said to be S-$JSR^*_{(p)}$ mappings if and only if each sequence $\{\mathrm{x_n}\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \text{ in } X$$

$$\Longrightarrow \max\{\alpha p(TSx_n,STx_n), \alpha p\ (STx_n,TSx_n)\}$$
$$\leq \max\{\alpha p(SSx_n,TTx_n), \alpha p(TTx_n\ ,SSx_n)\}$$
$$\text{where } \alpha = \lim\sup \text{ or } \lim\inf$$

Now we give some lemma which are useful in our main results.

Lemma 2.1 (see [3] and [13])

If (X,d)be a metric space, p be a w-distance on X, $\{x_n\}$, $\{y_n\} \subset X$ be sequences and $\{\alpha_n\},\{\beta_n\} \subset (0,\infty)$be sequences such that$\alpha_n \to 0$ and $\beta_n \to 0$ and for $x,y,z \in X$. Then we have the following conditions:

1. $p(x_n,y) \leq\ \alpha_n, p(x_n,z) \leq\ \beta_n, \forall\, n \in N \Longrightarrow y = z$.Particularly, if$p(x,y) = 0, p(x,z) = 0\ \Longrightarrow\ y = z$.
2. $p(x_n,y_n) \leq\ \alpha_n, p(x_n\ ,z) \leq\ \beta_n \forall\, n \in N \Longrightarrow y_n \to z$.
3. $p(x_n,x_m) \leq\ \alpha_n, \forall\, n,m \in N$ with $m > n \Longrightarrow \{x_n\}$ is a Cauchy sequence.
4. $p(y,x_n) \leq\ \alpha_n, \forall n \in N \Longrightarrow \{x_n\}$is a Cauchy sequence.

Lemma 2.2

If (X,d)be a metric space, p be a w-distance on $X$and let S and T be self mappings on X, satisfying $Tx_n = Sx_{n+1}$ for $n = 0,1,2,\ldots$,assume that there exists a continuous self mapping ξ of $[0,\infty]$ such that

$p(Tx,Ty) \leq \xi\big(p(Sx,Sy)\big)$

(2.2.1)

for all $x,y \in X$ and for each $t > 0$

$\xi(t) < t$

(2.2.2)

Then

(A) for an arbitrary $\epsilon > 0$, there exist positive integer m,s such that $m \leq n < s$ implies $p(Tx_n\ ,Tx_s) < \epsilon$.

(B) the sequence $\{Tx_n\}$ is a Cauchy sequence.

Proof

We have

$$p(Tx_n,Tx_{n+1}) \leq \xi\big(p(Sx_n,Sx_{n+1})\big)$$
$$= \xi\big(p(Tx_{n-1},Tx_n)\big)$$
$$< p(Tx_{n-1},Tx_n)$$

for$n = 1,2,3,\ldots$.. Thus $\{p(Tx_n,Tx_{n+1})\}$ is a decreasing sequence of non negative real number and there exists non negative real number λ such that

$$\lim_{n\to\infty}(n+1)p(Tx_n,Tx_{n+1}) = \lambda,$$

Let $\lambda > 0$, then the inequality

$$p(Tx_n,Tx_{n+1}) \leq \xi\big(p(Tx_{n-1},Tx_n)\big)$$

Now the continuity of ξ we have$\lambda < \xi(\lambda) < \lambda$,which is contradiction.

Therefore $\lambda = 0$ so $p(Tx_n,Tx_{n+1}) \to 0$ as $n \to \infty$.

(A) Now suppose that (A) does not hold. Then, there exists an $\epsilon > 0$ such that for all sufficiently large positive integer$k$, there exist positive integers $s_k,n_k$with $k \leq n_k < s_k$such that

$\epsilon \leq \big(p(Tx_{nk},Tx_{sk})\big),\big(p(Tx_{nk},Tx_{nk-1})\big) < \epsilon$

(2.2.3)

From (2.2.3), we have

$$p(Tx_{nk},Tx_{sk}) \to \epsilon \text{ and } p(Tx_{nk},Tx_{nk-1}) \to 0 \text{ as } k \to \infty$$

And$p(Tx_{nk},Tx_{sk}) \leq\ p(Tx_{nk},Tx_{nk+1}) + p(Tx_{nk+1},Tx_{sk})$

$$\leq\ p(Tx_{nk},Tx_{nk+1}) + \xi\big(p(Tx_{nk+1},Tx_{sk})\big)$$
$$\leq\ p(Tx_{nk},Tx_{nk+1}) + \xi\big(p(Tx_{nk},Tx_{sk-1})\big) \quad (2.2.4)$$

By the hypothesis and (2.2.4),we obtain $\epsilon \leq \xi(\epsilon) < \epsilon$. This is contradiction therefore (A) holds.

Alsowe have from the third condition of the definition of a w-distance $p$and(A) that $\{Tx_n\}$ is a Cauchy sequence.

Lemma 2.3

If (X,d)be a metric space, p be a w-distance on X, let S and T be self-mappings on Xsuch that$Tx_n = Sx_{n+1}$ for $\mathrm{n} = 0,1,2,\ldots\ldots$, with the following conditions: for given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq p(Sx,Sy) < \epsilon + \delta \Rightarrow p(Tx,Ty) < \epsilon, (2.3.1)$$

$p(Sx,Sy) <\in \Rightarrow p(Tx,Ty) \le 1/2\, p(Sx,Sy)$ (2.3.2)

The

(C) For an arbitrary $\in > 0$, there exists a positive integer M such that $M \le n < s$ implies $p(Tx_n, Tx_S) <\in$.

(D) The sequence $\{Tx_n\}$is a cauchy sequence.

3. Main Results

Theorem 3.1

If (X,d)be a metric space, p be a w-distance on $X$and let S and T be $S-JSR(p)$ self mappings of X, satisfying $T(X) \subset S(X)$, (2.2.1), (2.2.2) and for each $z \in X$ with $z \ne Tz$ or $z \ne Sz$

$$\inf\{p(Tx,z)+p(Sx,z)+p(STx,STx)+p(SSx,TTx), x \in X\} \text{ (3.1.1)}$$

Then there is a unique common fixed point of T and S.

Proof

Because$T(X) \subset S(X)$,therefore in X, we can define a sequence $\{x_n\}$such that $Tx_n = Sx_{n+1}$. Since X is complete and$Tx_n = Sx_{n+1}$there exists z in X such that $Tx_n \to z$ and$Sx_n \to z$.

Suppose that $z \ne Tz$or $z \ne Sz$, since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$, therefore by (A) and the lower semi continuity, we have

$$\lim_{n\to\infty} p(Tx_n, z) = \lim_{n\to\infty} p(Sx_n, z)$$

Now,

$$\begin{aligned} 0 &< \inf\{p(Tx,z)+p(Sx,z)+p(TSx,TSx) + p(SSx,TTx), x \in X\} \\ &\le \inf\{p(Tx_n,z)+p(Sx_n,z)+p(TSx_n,TSx_n) + p(SSx_n,TTx_n)\} \\ &\le \inf\{p(Tx_n,z)+p(Sx_n,z) + max[\alpha p(STx_n,TSx_n), \alpha p(SSx_n,TTx_n)] + p(SSx_n,TTx_n)\} < 0. \end{aligned}$$

which is a contradiction and hence, our assumption that $z \ne Tz$ or $z \ne Sz$ was wrong. Therefore,$Tz = Sz = z$. Applying (2.2.1)of Lemma 2.2 and (2.3.1), (2.3.2) of Lemma 2.3 uniqueness of the fixed point is obvious.

Theorem 3.2

Let(X,d) be a complete metric space with a w-distance p and let S and T be S-JSR*(p) self mappings of X, satisfying $T(X) \subset S(X)$, (2.2.1and (2.2.2), for each $z \in X$ with $z \ne Tz$ or $z \ne Sz$

$$\inf\{p(Tx,z)+p(Sx,z)+p(TSx,STx)+p(SSx,TTx), x \in X\} \text{(3.2.1)}$$

Then there is a unique common fixed point of T and S.

Proof

Because $T(X) \subset S(X)$, therefore in X, we can define a sequence $\{x_n\}$such that $Tx_n = Sx_{n+1}$. Since X is complete and$Tx_n = Sx_{n+1}$there exists $z \: in \: X$ such that $Tx_n \to z \: and \: Sx_n \to z$.

Suppose that $z \ne Tz$or $z \ne Sz$, since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$, therefore by (A) and the lower semi continuity, we have

$$\lim_{n\to\infty} p(Tx_n, z) = \lim_{n\to\infty} p(Sx_n, z)$$

Now,

$$\begin{aligned} 0 &< \inf\{p(Tx,z)+p(Sx,z)+p(TSx,TSx) + p(SSx,TTx), x \in X\} \\ &\le \inf\{p(Tx_n,z)+p(Sx_n,z)+p(TSx_n,TSx_n) + p(SSx_n,TTx_n)\} \\ &\le \inf\{p(Tx_n,z)+p(Sx_n,z) + max[\alpha p(STx_n,TSx_n), \alpha p(SSx_n,TTx_n)] + p(SSx_n,TTx_n)\} < 0. \end{aligned}$$

which is a contradiction and hence, our assumption that $z \ne Tz$ or $z \ne Sz$ was wrong. Therefore $Tz = Sz = z$. Applying (2.2.1) of Lemma 2.2 and (2.3.1), (2.3.2) of Lemma 2.3 uniqueness of the fixed point is obvious.

4. Examples

Example 4.1

Let $X = [0,1]$ with $d(x,y) = |x-y|$ and S,T are two self mapping on X defined by $Sx = \frac{2}{x+2}, Tx = \frac{1}{x+1}$for $x \in X$. Now we have the sequence $\{x_n\}$ in X is defined as $x_n = \frac{1}{n}$, $n \in N$.Then we have

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1$$

$|STx_n - Tx_n| \to \frac{1}{3}and|SSx_n - Sx_n| \to \frac{2}{3}$ as $n \to \infty$.

Clearly we have

$|STx_n - Tx_n| < |SSx_n - Sx_n|$.

Thus pair (S,T) is S-JSR mapping. But this pair is neither compatible nor weakly compatible nor other non commuting mapping. Hence pair of JSR mapping is more general than others.

Example 4.2

Let $X = [0,1]$ with $p(x,y) = \max\{|\frac{x}{2} - y, 12|x-y|\}$ and S,T are two self mapping on X defined by

$Sx = \frac{2}{x+2}, Tx = \frac{1}{x+1}$ for $x \in X$.

Now we have the sequence $\{x_n\}$ in X is defined as$x_n = \frac{1}{n}$, $n \in N$.Then we have

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1$. Now

$$p(STx_n, Tx_n) = \max\left\{\left|\frac{STx_n}{2} - Tx_n\right|, \frac{1}{2}|STx_n - Tx_n|\right\} = \max\left\{\frac{2}{3}, \frac{1}{6}\right\} = \frac{2}{3}$$

$$p(Tx_n, STx_n) = \max\left\{\left|\frac{Tx_n}{2} - Tx_n\right|, \frac{1}{2}|Tx_n - Tx_n|\right\} = \max\left\{\frac{1}{6}, \frac{1}{6}\right\} = \frac{1}{6}$$

$$p(SSx_n, Sx_n) = \max\left\{\left|\frac{SSx_n}{2} - Sx_n\right|, \frac{1}{2}|SSx_n - Sx_n|\right\} = \max\left\{\frac{2}{3}, \frac{1}{3}\right\} = \frac{2}{3}$$

$$p(Sx_n, SSx_n) = \max\left\{\left|\frac{Sx_n}{2} - SSx_n\right|, \frac{1}{2}|Sx_n - SSx_n|\right\} = \max\left\{\frac{1}{6}, \frac{1}{3}\right\} = \frac{1}{3}$$

Clearly pair (S,T) is S-JSR(p) mapping. Also $p(x,y) \ne p(y,x)$.

Example 4.3

Let $X = [0,1]$ with $p(x,y) = \max\{|\frac{x}{2} - y, 12|x-y|\}$ and S,T are two self mapping on X defined by

$Sx = \frac{2}{x+2}, Tx = \frac{1}{x+1}$ for $x \in X$.

Now we have the sequence $\{x_n\}$ in X is defined as $x_n = 1 - \frac{1}{n}$, $n \in N$.Then we have

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1.$$

In view of Theorem 3.1, $z = 1$ is unique common fixed point of $T$and .

Conclusion

So we have established two fixed point theorems for non-commutingJSR and JSR*mappingsvia w-distance in complete metric space are proved supported with examples.

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